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Direct Limits of Effect Algebras†

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In this paper, we prove that direct limits exist in the category of effect algebras and effect algebra-morphisms. Then, as a consequence, we obtain similar known results for the categories of orthomodular posets and orthomodular lattices.

1. INTRODUCTION

The effects of a quantum mechanical system $\mathcal{S}$ can be represented by self-adjoint operators A on a separable complex Hilbert space $\mathcal{H}$ such that $O \leq A \leq I$, where O and I are the zero and the identity operators, respectively, on $\mathcal{H}$ [6]. The set $\mathcal{E}(\mathcal{H})$ of all such operators A forms an (ordered) algebraic structure which is the prototypical example of the effect algebras (and difference posets) discussed in this paper [6, 14], and it provides a mathematical model for the study of unsharp quantum logics [6]. Furthermore, effect algebras generalize the various ordered structures that have been used as frameworks in the quantum logic approach to the foundation of quantum physics which was originated about 60 years ago by Birkhoff and von Neumann [2], who proposed the framework of a modular, complemented lattice. This framework was later generalized to orthomodular lattices and posets [1, 9, 10, 13, 15–17], and most recently to orthoalgebras [7, 10–12].

Direct and inductive limits of orthomodular lattices and posets were considered by Fischer and Rüttimann [5] and by Dacey [4], who showed their connection and importance to the study of operational statistics and quantum field theory [8, 9, 19, 16]. In this paper, we shall study direct limits of effect algebras. By choosing suitable morphisms, effect algebras form a (concrete) category. We shall prove, by construction, that direct limits in such

† This paper is dedicated to the memory of Prof. Gottfried T. Rüttimann.

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a category exist (see Theorems 3.1 and 3.14). Then, as a consequence, we obtain the result (see Corollary 3.3) that such a direct limit is an orthomodular poset if each of the effect algebras of the directed system is an orthomodular poset, and the result (see Theorem 3.9) that such a direct limit is an orthomodular lattice if each of the effect algebras of the direct system is an orthomodular lattice and if each of the effect algebra-monomorphisms preserves disjointness (see Definition 3.6). This last result is an improvement of a corresponding result of Dacey [4] and of Fischer and Rüttimann [5], where it is assumed that the effect algebra-monomorphisms are residuated (see Definition 3.10). We mention that similar results for difference posets have recently been obtained by S. Pulmannovà [18].

2. DEFINITIONS AND PRELIMINARIES

The following definition was introduced by Foulis and Bennett [6].

Definition 2.1. An *effect algebra* (abbreviated EA) is a system $(L, \oplus, 0, 1)$ consisting of a set L containing two special elements 0, 1 and equipped with a partially defined binary operation $\oplus$ satisfying the following conditions $\forall a, b, c \in L$:

- (EA1) (*Commutative law*) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.
- (EA2) (*Associative law*) If $b \oplus c$ is defined and $a \oplus (b \oplus c)$ is defined, then $a \oplus b$ is defined, $(a \oplus b) \oplus c$ is defined, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.
- (EA3) (*Orthocomplementation law*) For every $a \in L$ there exists a unique $b \in L$ such that $a \oplus b$ is defined and $a \oplus b = 1$.
- (EA4) (Zero–one law) If $1 \oplus a$ is defined, then $a = 0$.

Let $L = (L, \oplus, 0, 1)$ be an effect algebra and $a, b \in L$. Following [6], we say that a is *orthogonal* to b in L and write $a \perp b$ if and only if $a \oplus b$ is defined in L. We define $a \leq b$ to mean that there exists $c \in L$ such that $a \perp c$ and $b = a \oplus c$. The unique element $b \in L$ corresponding to a in Condition (EA3) is called the *orthocomplement* of a and is written as $a' := b$. For any effect algebra L, it can be easily proved [6] that $0 \leq a \leq 1$ holds for all $a \in L$, that $a \perp b$ iff $a \leq b'$, that, with $\leq$ as defined above, $(L, \leq, 0, 1)$ is a partially ordered set (poset), and that L satisfies the so-called *orthomodular identity* (OMI):

$$\forall a, b \in L, \quad a \leq b \Rightarrow b = a \oplus (a \oplus b')'$$

For $a, b, c \in L$, we write $c = a \vee b$ (resp., $c = a \wedge b$) to indicate that c is the least upper bound (resp., greatest lower bound) of a and b in the poset $(L, \leq, 0, 1)$.

Definition 2.2. An *orthoalgebra* [7, 10, 12] is an effect algebra L in which the zero–one law is replaced by the stronger condition: $a \in L$, $a \oplus a$ defined $\Rightarrow a = 0$. Recall that an *orthomodular poset* (OMP) [10] may be regarded as an orthoalgebra L that satisfies the following additional condition [7]: $a, b \in L$, $a \perp b \Rightarrow a \vee b$ exists and $a \vee b = a \oplus b$. An *orthomodular lattice* (OML) may be defined as an OMP which is also a lattice.

Definition 2.3. Let L and Q be EAs. A mapping $\phi\colon L \to Q$ is called an *EA-morphism* iff (i) $\phi(1) = 1$ and (ii) for $a, b \in L$, $a \perp b \Rightarrow \phi(a) \perp \phi(b)$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$. Following [11], an EA-morphism $\phi\colon L \to Q$ is called *special* if it satisfies the following condition:

$$u, v \in \phi(L), \quad u \perp v \Rightarrow \exists\, a, b \in L \quad \text{with} \quad a \perp b,$$
$$\phi(a) = u, \quad \text{and} \quad \phi(b) = v$$

An EA-morphism $\phi\colon L \to Q$ is called a *monomorphism* if it is special and there is an EA-morphism $\psi\colon \phi(L) \to L$ such that $\psi\phi = \mathrm{id}_L$, where id_L denotes the identity mapping on L.

It can be easily checked [11] that if $\phi\colon L \to Q$ is an EA-morphism, then the following hold: (a) $\phi(0) = 0$, (b) $\phi(a') = \phi(a)'$ $\forall a \in L$, and (c) for $a, b \in L$, $a \leq b \Rightarrow \phi(a) \leq \phi(b)$. Furthermore, it can be shown [11, Theorem 2. 6] that an EA-morphism $\phi\colon L \to Q$ is a monomorphism iff $\forall a, b \in L$, $a \perp b$ in $L \Leftrightarrow \phi(a) \perp \phi(b)$ in Q. For more about EA-morphisms (which are the same as orthoalgebra-morphisms), the reader may consult [11].

Throughout the paper, we let $\mathscr{E}$ denote the category with effect algebras as objects and with EA-monomorphisms as morphisms.

Definition 2.4. A *directed system* in the category $\mathscr{E}$ is a pair $(\{L_\alpha\}_{\alpha \in D}; \{\phi^\alpha_\beta\}_{\alpha \leq \beta})$, where (i) D is a directed set, (ii) L_α is an EA $\forall \alpha \in D$, (iii) if $\alpha, \beta \in D$, $\alpha \leq \beta$, then $\phi^\alpha_\beta\colon L_\alpha \to L_\beta$ is an EA-monomorphism, (iv) $\phi^\beta_\gamma \phi^\alpha_\beta = \phi^\alpha_\gamma$ for $\alpha \leq \beta \leq \gamma$, and (v) $\phi^\alpha_\alpha = \mathrm{id}_{L_\alpha}$.

A *direct* (or *inductive*) *limit* of a directed system $(\{L_\alpha\}_{\alpha \in D}; \{\phi^\alpha_\beta\}_{\alpha \leq \beta}) \in \mathscr{E}$ is a pair $(L; \{\phi^\alpha\}_{\alpha \in D}) \in \mathscr{E}$, where L is an EA and each $\phi^\alpha\colon L_\alpha \to L$ is an EA-monomorphism such that (a) $\phi^\beta \phi^\alpha_\beta = \phi^\alpha$ for $\alpha \leq \beta$ and (b) if $\psi_\alpha\colon L_\alpha \to Q$, where Q is an EA and ψ_α is an EA-monomorphism, are given such that $\psi_\beta \phi^\alpha_\beta = \psi_\alpha$ for $\alpha \leq \beta$, then there exists a unique EA-monomorphism $\psi\colon L \to Q$ such that $\psi_\alpha = \psi\phi^\alpha$ for $\alpha \in D$.

In the sequel, we let $(\{L_\alpha\}_{\alpha \in D}; \{\phi^\alpha_\beta\}_{\alpha \leq \beta})$ be a fixed directed system in the category $\mathscr{E}$ with $L_\alpha \cap L_\beta = \emptyset$ if $\alpha \neq \beta$, and we let $X := \cup_{\alpha \in D} L_\alpha$.

Definition 2.5. Define a relation $\sim$ on X by: $x \sim y$, $x \in L_\alpha$, $y \in L_\beta$, iff $\exists \gamma \in D$ such that $\alpha, \beta \leq \gamma$ and $\phi^\alpha_\gamma x = \phi^\beta_\gamma y$.

It is easy to see that $\sim$ is an equivalence relation on X, and if $x \in L_\alpha$, $y \in L_\beta$ and $x \sim y$, then for any $\gamma \in D$ such that $\alpha, \beta \leq \gamma$ we have $\phi_\gamma^\alpha x = \phi_\gamma^\beta y$.

Definition 2.6. Let $\bar{x} := \{y \in X: y \sim x\}$, and let $L := \{\bar{x}: x \in X\}$. Define a partial binary operation $\oplus$ on L by: $\bar{x} \oplus \bar{y}$ is defined in L iff $\exists \alpha \in D$, $x_\alpha \in \bar{x} \cap L_\alpha$ and $y_\alpha \in \bar{y} \cap L_\alpha$ such that $x_\alpha \oplus y_\alpha$ is defined in L_α. In this case, we shall write $\bar{x} \oplus \bar{y} := \overline{x_\alpha \oplus y_\alpha}$.

Lemma 2.7. If $\bar{x} \oplus \bar{y}$ is defined in L, $x_\beta \in \bar{x} \cap L_\beta$ and $y_\beta \in \bar{y} \cap L_\beta$, then $x_\beta \oplus y_\beta$ is defined in L_β and $\bar{x} \oplus \bar{y} = \overline{x_\beta \oplus y_\beta}$. In particular, $\oplus$ as defined above is well defined.

Proof. Since $\bar{x} \oplus \bar{y}$ is defined in L, $\exists \alpha \in D$, $x_\alpha \in \bar{x} \cap L_\alpha$ and $y_\alpha \in \bar{y} \cap L_\alpha$ such that $x_\alpha \oplus y_\alpha$ is defined in L_α. Choose $\gamma \in D$ such that $\alpha, \beta \leq \gamma$. Then $\phi_\gamma^\alpha x_\alpha \oplus \phi_\gamma^\alpha y_\alpha$ is defined in L_γ. Since $x_\alpha \sim x_\beta$, $y_\alpha \sim y_\beta$, and $\alpha, \beta \leq \gamma$, we have $\phi_\gamma^\alpha x_\alpha = \phi_\gamma^\beta x_\beta$ and $\phi_\gamma^\alpha y_\alpha = \phi_\gamma^\beta y_\beta$. Hence $\phi_\gamma^\alpha(x_\alpha \oplus y_\alpha) = \phi_\gamma^\alpha x_\alpha \oplus \phi_\gamma^\alpha y_\alpha = \phi_\gamma^\beta x_\beta \oplus \phi_\gamma^\beta y_\beta = \phi_\gamma^\beta(x_\beta \oplus y_\beta)$, which implies that $x_\beta \oplus y_\beta$ is defined in L_β (since ϕ_γ^β is an EA-monomorphism) and $x_\alpha \oplus y_\alpha \sim x_\beta \oplus y_\beta$. Therefore $\bar{x} \oplus \bar{y} = \overline{x_\beta \oplus y_\beta}$. ■

The proof of the following lemma is straightforward.

Lemma 2.8. 1. If $x \in L_\alpha$, $y \in L_\beta$, and $x \sim y$, then $x' \sim y'$.

2. If $x_\alpha, y_\alpha \in L_\alpha$ and $x_\alpha \sim y_\alpha$, then $x_\alpha = y_\alpha$.

3. $0_\alpha \sim 0_\beta$, where 0_α and 0_β are the least elements in L_α and L_β, respectively.

4. $1_\alpha \sim 1_\beta$, where 1_α and 1_β are the greatest elements in L_α and L_β, respectively.

Moreover, $\overline{0_\alpha}$ and $\overline{1_\alpha}$, $\alpha \in D$, are the least and the greatest elements in L, respectively.

Note that Lemma 2.8 justifies the following definition.

Definition 2.9. Let $1 := \overline{1_\alpha}$, $\alpha \in D$, and $0 := \overline{0_\alpha}$, $\alpha \in D$.

Using Definitions 2.6, 2.9, and Lemmas 2.7, 2.8, it is easy to prove the following result.

Lemma 2.10. $(L, \oplus, 0, 1)$ is an effect algebra.

3. RESULTS

Now we are ready to establish the main result of this article.

Theorem 3.1. The direct limit of any directed system of effect algebras exists. More precisely, if $(\{L_\alpha\}_{\alpha \in D}; \{\phi_\beta^\alpha\}_{\alpha \leq \beta})$ is a directed system in the category $\mathscr{E}$, then its direct limit exists in the same category and equals $(L, \{\phi^\alpha\}_{\alpha \in D})$ where L is as defined in Definition 2.6 and $\phi^\alpha: L_\alpha \to L$ is defined by $\phi^\alpha x_\alpha := \overline{x_\alpha}$.

Proof. Let $(\{L_\alpha\}_{\alpha \in D}; \{\phi^\alpha_\beta\}_{\alpha \leq \beta})$ be a directed system in $\mathscr{E}$ and let $(L, \oplus, 0, 1)$ be as in Definitions 2.6 and 2.9. Then by Lemma 2.10, L is an EA. By the definition of ϕ^α we have $\phi^\alpha(x_\alpha \oplus y_\alpha) = \overline{x_a \oplus y_\alpha} = \overline{x_\alpha} \oplus \overline{y_\alpha} = \phi^\alpha(x_\alpha) \oplus \phi^\alpha(y_\alpha)$, $\phi^\alpha(1_\alpha) = \overline{1_\alpha} = 1$, and, if $\phi^\alpha x_\alpha \perp \phi^\alpha y_\alpha$ in L, then $\phi^\alpha x_\alpha \oplus \phi^\alpha y_\alpha = \overline{x_\alpha} \oplus \overline{y_\alpha}$ is defined in L, and hence, by Lemma 2.7, $x_\alpha \oplus y_\alpha$ is defined in L_α. Thus $x_\alpha \perp y_\alpha$ in L_α and therefore, by the remarks following Definition 2.3, ϕ^α is an EA-monomorphism.

Next, we shall show that the EA-monomorphisms ϕ^α, $\alpha \in D$, satisfy conditions (a) and (b) of Definition 2.4. First, note that $\phi^\alpha_\beta x_\alpha \sim x_\alpha$ for all $\alpha \leq \beta$, since $\phi^\beta_\gamma \phi^\alpha_\beta x_\alpha = \phi^\alpha_\gamma x_\alpha$ $\forall \gamma \in D$ with $\alpha, \beta \leq \gamma$. It follows that for all $x_\alpha \in L_\alpha$ and for all $\alpha \leq \beta$, we have $\phi^\beta \phi^\alpha_\beta x_\alpha = \overline{\phi^\alpha_\beta x_\alpha} = \overline{x_\alpha} = \phi^\alpha x_\alpha$. Therefore $\phi^\beta \phi^\alpha_\beta = \phi^\alpha$ $\forall \alpha \leq \beta$.

Second, suppose that there exists $(Q; \{\psi_\alpha\}_{\alpha \in D})$ in the same category $\mathscr{E}$ such that $\psi_\beta \phi^\alpha_\beta = \psi_\alpha$ for all $\alpha \leq \beta$. Define $\psi: L \to Q$ by $\psi \overline{x_\alpha} := \psi_\alpha x_\alpha$, $\alpha \in D$. Then ψ is well defined, since, if $x_\beta \in \overline{x_\alpha} \cap L_\beta$, then $\exists \gamma \in D$ such that $\alpha, \beta \leq \gamma$ and $\phi^\beta_\gamma x_\beta = \phi^\alpha_\gamma x_\alpha$, which implies that $\psi_\beta x_\beta = \psi_\gamma \phi^\beta_\gamma x_\beta = \psi_\gamma \phi^\alpha_\gamma x_\alpha = \psi_\alpha x_\alpha$. Also, we have $\psi(\overline{x_\alpha} \oplus \overline{y_\alpha}) = \psi(\overline{x_\alpha \oplus y_\alpha}) = \psi_\alpha(x_\alpha \oplus y_\alpha) = \psi_\alpha x_\alpha \oplus \psi_\alpha y_\alpha = \psi \overline{x_\alpha} \oplus \psi \overline{y_\alpha}$, $\psi(1_L) = \psi(\overline{1_\alpha}) = \psi_\alpha(1_\alpha) = 1_Q$, and if $\psi \overline{x_\alpha} \perp \psi \overline{y_\alpha}$ in Q, then $\psi \overline{x_\alpha} \oplus \psi \overline{y_\alpha} = \psi_\alpha x_\alpha \oplus \psi_\alpha y_\alpha = \psi_\alpha(x_\alpha \oplus y_\alpha)$ is defined in Q, which implies that $x_\alpha \oplus y_\alpha$ is defined in L_α and this implies that $\bar{x}_\alpha \oplus \bar{y}_\alpha$ is defined in L, by Definition 2.6. Thus ψ is an EA-monomorphism.

Finally, if there is an EA-monomorphism $\phi: L \to Q$ such that $\phi \phi^\alpha = \psi_\alpha$ $\forall \alpha \in D$, then $\phi \overline{x_\alpha} = \phi \phi^\alpha x_\alpha = \psi_\alpha x_\alpha = \psi \overline{x_\alpha}$ $\forall \alpha \in D$. Thus, ψ is the unique EA-monomorphism such that $\psi \phi^\alpha = \psi_\alpha$ $\forall \alpha \in D$. ■

Before we derive some consequences of Theorem 3.1, we need to establish a few more lemmas.

Lemma 3.2. Suppose that each L_α, $\alpha \in D$, is an OMP. If $\bar{x} \perp \bar{y}$ in L, then $\bar{x} \vee \bar{y}$ exists in L and $\bar{x} \vee \bar{y} = \bar{x} \oplus \bar{y}$.

Proof. The proof is straightforward and therefore it is omitted. ■

Corollary 3.3. The direct limit of a directed system of effect algebras is an orthomodular poset if each of the effect algebras in the directed system is an orthomodular poset.

Proof. It follows from Theorem 3.1, Lemma 3.2, and the fact [7, Theorem 2.11] that an EA in which the join of every orthogonal pair exists is an OMP. ■

Remark 3.4. Theorem 3.1 and Corollary 3.3 contain the result [5, part III.B] that direct limits exist in the category of orthomodular posets and orthomodular poset-monomorphisms.

Although the proof of the following lemma appears in [6, Theorem 6.6], the proof given here was independently obtained by the author while working on an earlier version of this paper with Prof. R. J. Greechie at Louisiana Tech University during the spring of 1993.

Lemma 3.5. Let L_1 and L_2 be OMLs, and let $\phi: L_1 \to L_2$ be an EA-morphism. Then ϕ is an OML-morphism iff $x, y \in L_1$, $x \wedge y = 0 \Rightarrow \phi(x) \wedge \phi(y) = 0$.

Proof. If ϕ is an OML-morphism, then the claimed implication trivially holds. Conversely, suppose that $\forall x, y \in L_1 \ni x \wedge y = 0$ we have $\phi(x) \wedge \phi(y) = 0$. We claim that $\phi(a \wedge b) = \phi(a) \wedge \phi(b)$ $\forall a, b \in L_1$. Indeed, it is clear that $\phi(a \wedge b) \leq \phi(a) \wedge \phi(b)$ for $a, b \in L_1$. To show equality, it is enough to show, using the orthomodular identity, that $\phi(a) \wedge \phi(b) \wedge (\phi(a \wedge b))' = 0$. By the orthomodular identity, we see that $a' \leq a' \vee b' \Rightarrow \phi(a' \vee b') = \phi(a') \vee \phi((a' \vee b') \wedge a)$, and that $b' \leq a' \vee b' \Rightarrow \phi(a' \vee b') = \phi(b') \vee \phi((a' \vee b') \wedge b)$. Hence, using the hypothesis, we have

$$\begin{aligned}
\phi(a) \wedge \phi(b) \wedge (\phi(a \wedge b))' &= (\phi(a) \wedge \phi(a' \vee b')) \wedge (\phi(b) \wedge \phi(a' \vee b')) \\
&= \phi(a) \wedge (\phi(a') \vee \phi((a' \vee b') \wedge a)) \wedge \phi(b) \\
&\quad \wedge (\phi(b') \vee \phi((a' \vee b') \wedge b)) \\
&= \phi(a) \wedge \phi((a' \vee b') \wedge a) \wedge \phi(b) \\
&\quad \wedge \phi((a' \vee b') \wedge b) \\
&= \phi((a' \vee b') \wedge a) \wedge \phi\,((a' \vee b') \wedge b) = 0
\end{aligned}$$

since $(a' \vee b') \wedge a \wedge (a' \vee b') \wedge b = (a' \vee b') \wedge (a \wedge b) = 0$. This proves the claim. Now since ϕ preserves orthocomplementation, the above claim and the De Morgan law imply that ϕ preserves joins. Therefore, ϕ is an OML-morphism. ■

Definition 3.6. We say that a morphism $\phi: L \to Q$ of EAs *preserves disjointness* if $x \wedge y = 0$ in $L \Rightarrow \phi(x) \wedge \phi(y) = 0$ in Q.

The proof of the following lemma follows from Lemma 3.5 and the remark following Definition 2.5.

Lemma 3.7. Assume each L_α, $\alpha \in D$, is an OML, and each ϕ^α_β, $\alpha, \beta \in D$, $\alpha \leq \beta$ preserves disjointness. If $x_\alpha, y_\alpha \in L_\alpha$ and $x_\beta, y_\beta \in L_\beta$, and if $x_\alpha \sim x_\beta$ and $y_\alpha \sim y_\beta$, then $x_\alpha \vee y_\alpha \sim x_\beta \vee y_\beta$.

Lemma 3.8. Assume that each L_α, $\alpha \in D$, is an OML and each ϕ^α_β, α, $\beta \in D$, $\alpha \leq \beta$, preserves disjointness. Then $\bar{x} \vee \bar{y}$ exists in L for all $\bar{x}, \bar{y} \in L$.

Proof. Let $\bar{x}, \bar{y} \in L$. There exist $\alpha, \beta \in D$ such that $\overline{x_\alpha} = \bar{x}$ and $\overline{y_\beta} = \bar{y}$. Choose $\gamma \in D$ such that $\alpha, \beta \leq \gamma$. Then $\exists x_\gamma, y_\gamma \in L_\gamma$ such that $\overline{x_\gamma} = \overline{x_\alpha}$ and $\overline{y_\gamma} = \overline{y_\beta}$. Since L_γ is an OML, $x_\gamma \vee y_\gamma$ exists in L_γ. Since $x_\gamma, y_\gamma \leq x_\gamma \vee y_\gamma$, we have $\bar{x}_\gamma, \bar{y}_\gamma \leq \overline{x_\gamma \vee y_\gamma}$.

Now suppose that $\exists \bar{u} \in L$, $\overline{x_\gamma}, \overline{y_\gamma} \leq \bar{u}$. Then $\exists \delta_1 \in D$, $\overline{u_{\delta_1}} = \bar{u}$. Choose $\delta_2 \in D$, $\delta_1, \gamma \leq \delta_2$. Then $\exists x_{\delta_2}, y_{\delta_2}, u_{\delta_2} \in L_{\delta_2}$ such that $\overline{x_{\delta_2}} = \overline{x_\gamma}$, $\overline{y_{\delta_2}} = \overline{y_\gamma}$, and $\overline{u_{\delta_2}} = \overline{u_{\delta_1}} = \bar{u}$. Thus $\overline{x_{\delta_2}}, \overline{y_{\delta_2}} \leq \overline{u_{\delta_2}}$. This implies that there exist $\delta \in D$ and $x_\delta, y_\delta, u_\delta \in L_\delta$ such that $\overline{x_\delta} = \overline{x_{\delta_2}}$, $\overline{y_\delta} = \overline{y_{\delta_2}}$, $\overline{u_\delta} = \overline{u_{\delta_2}}$, and $x_\delta, y_\delta \leq u_\delta$. Hence $x_\delta \vee y_\delta \leq u_\delta$, which implies that $\overline{x_\delta \vee y_\delta} \leq \overline{u_\delta}$. By Lemma 3.7, since $x_\delta \sim x_{\delta_2} \sim x_\gamma$ and $y_\delta \sim y_{\delta_2} \sim y_\gamma$, we have $\overline{x_\gamma \vee y_\gamma} = \overline{x_{\delta_2} \vee y_{\delta_2}} = \overline{x_\delta \vee y_\delta} \leq \overline{u_\delta} = \overline{u_{\delta_2}} = \bar{u}$. Therefore, since $\overline{x_\gamma} = \bar{x}$ and $\overline{y_\gamma} = \bar{y}$, $\bar{x} \vee \bar{y}$ exists in L and $\bar{x} \vee \bar{y} = \overline{x_\gamma \vee y_\gamma}$. ■

As a consequence of Corollary 3.3, Lemma 3.5, and Lemma 3.8, we obtain the following main result.

Theorem 3.9. Let $(\{L_\alpha\}_{\alpha \in D}; \{\phi^\alpha_\beta\}_{\alpha \leq \beta})$ be a directed system in $\mathscr{E}$ such that each ϕ^α_β preserves disjointness, and let $(L, \{\phi^\alpha\}_{\alpha \in D})$ be its direct limit. Assume that each L_α is an OML. Then L is an OML and each ϕ^α preserves disjointness; hence $(L, \{\phi^\alpha\}_{\alpha \in D}) \in \mathscr{L}$, the category of OMLs with their monomorphisms.

Proof. We need only show that each ϕ^α preserves disjointness, since the other assertions follow from the above-mentioned results. To this end, fix $\alpha \in D$ and let $x_\alpha, y_\alpha \in L_\alpha$ be such that $x_\alpha \wedge y_\alpha = 0$. We claim that $\phi^\alpha x_\alpha \phi^\alpha y_\alpha$exists in L and equals 0. To see this, suppose that $\exists l \in L \ni \phi^\alpha x_\alpha$, $\phi^\alpha y_\alpha \geq l$ in L. Then $\exists \beta \geq \alpha \ni l = \phi^\beta(z_\beta)$ for some $z_\beta \in L_\beta$. As $\phi^\alpha = \phi^\beta \phi^\alpha_\beta$, we get $\phi^\beta \phi^\alpha_\beta x_\alpha$, $\phi^\beta \phi^\alpha_\beta y_\alpha \geq \phi^\beta(z_\beta)$. Since ϕ^β is an EA-monomorphism, this implies that $\phi^\alpha_\beta x_\alpha$, $\phi^\alpha_\beta y_\alpha \geq z_\beta$ and hence z_β is a lower bound of $\{\phi^\alpha_\beta x_\alpha, \phi^\alpha_\beta y_\alpha\}$ in L_β. Now, by Lemma 3.5, ϕ^α_β is an OML-morphism, since it preserves disjointness; so we have $0 = \phi^\alpha_\beta(x_\alpha \wedge y_\alpha) = \phi^\alpha_\beta x_\alpha \wedge \phi^\alpha_\beta y_\alpha$. This yields that $z_\beta = 0$ and therefore $l = \phi^\beta(z_\beta) = 0$. This proves the claim, and hence the theorem. ■

Definition 3.10. Let L and P be posets. A map $\phi\colon L \to P$ is called *monotone* if $x \leq y$ in L implies that $\phi(x) \leq \phi(y)$ in P. A monotone map $\phi\colon L \to P$ is called *residuated* provided that there exists a monotone map $\psi\colon P \to L$ such that $\psi\phi(x) \geq x\ \forall x \in L$ and $\phi\psi(p) \leq p\ \forall p \in P$.

Note that for each residuated map $\phi\colon L \to P$ there is a unique ψ as above. This unique ψ is called the *residual* of ϕ. It is well known from the theory of residuated maps [3] that a residuated map preserves all existing joins. Moreover, we have the following result.

Lemma 3.11. Every residuated EA-morphism preserves disjointness.

Proof. Let L and P be EAs and let $\phi\colon L \to P$ be a residuated EA-morphism. Let $a, b \in L$ with $a \wedge b = 0$. Then by the De Morgan law $a' \vee b'$ exists in L and equals $(a \wedge b)' = 1$. Since ϕ is residuated, $\phi(a' \vee b') = \phi(a') \vee \phi(b')$. Since ϕ is an EA-morphism, $\phi(1) = 1$, $\phi(a') = \phi(a)'$, and $\phi(b') = \phi(b)'$. Thus $1 = \phi((a \wedge b)') = \phi(a' \vee b') = \phi(a)' \vee \phi(b)'$, and so, by the De Morgan law again, $\phi(a) \wedge \phi(b) = 0$. ∎

As a consequence of Theorem 3.9 and Lemma 3.11, we obtain the following result, which is due to Dacey [4] and Fischer and Rüttimann [5].

Corrollary 3.12. Let $(\{L_\alpha\}_{\alpha \in D}; \{\phi_\beta^\alpha\}_{\alpha \leq \beta})$ be a directed system in $\mathscr{L}$ with residuated morphisms ϕ_β^α, and let $(L, \{\phi^\alpha\}_{\alpha \in \mathrm{D}})$ be its direct limit. Then $(L, \{\phi_\alpha\}_{\alpha \in D}) \in \mathscr{L}$. Moreover, the morphisms ϕ^α, $\alpha \in D$, are residuated.

Proof. That $(L, \{\phi^\alpha\}_{\alpha \in D}) \in \mathscr{L}$ follows immediately from Theorem 3.9 and Lemma 3.11. The proof that the ϕ^α are residuated can be found in [5, pp. 150–151]. ∎

Before we close this section, we would like to mention that Theorem 3.1 can be improved as follows. But, first, in the notation of Section 2, we have the following

3.13 Lemma. $\bar{x} \oplus \bar{y}$, $x \in L_\alpha$, $y \in L_\beta$, is defined in L (as in Definition 2.6) iff there exists $\gamma \geq \alpha, \beta$ such that $\phi_\gamma^\alpha x \oplus \phi_\gamma^\beta y$ is defined in L_γ.

Proof. ($\Rightarrow$): Assume that $\bar{x} \oplus \bar{y}$ is defined in L, where $x \in L_\alpha$, $y \in L_\beta$. Then, by Definition 2.6, there exist $\delta \in D$, $x_\delta \in \bar{x} \cap L_\delta$ and $y_\delta \in \bar{y} \cap L_\delta$ such that $x_\delta \oplus y_\delta$ is defined in L_δ. Since $x_\delta \sim x$, $\exists \gamma_1 \geq \delta, \alpha \ni \phi_{\gamma_1}^\delta x_\delta = \phi_{\gamma_1}^\alpha x$, and since $y_\delta \sim y$, $\exists \gamma_2 \geq \delta, \beta \ni \phi_{\gamma_2}^\delta y_\delta = \phi_\gamma^\delta y$. Choose $\gamma \geq \delta, \gamma_1, \gamma_2$. Then

$$\phi_\gamma^\alpha x = \phi_\gamma^{\gamma_1}\phi_{\gamma_1}^\alpha x = \phi_\gamma^{\gamma_1}\phi_{\gamma_1}^\delta x_\delta = \phi_\gamma^\delta x_\delta \quad \text{and} \quad \phi_\gamma^\beta y = \phi_\gamma^{\gamma_2}\phi_{\gamma_2}^\beta y = \phi_\gamma^{\gamma_2}\phi_{\gamma_2}^\delta y_\delta = \phi_\gamma^\delta y_\delta.$$

Now since $x_\delta \oplus y_\delta$ is defined in L_δ and ϕ_γ^δ is a morphism, we get that $\phi_\gamma^\delta(x_\delta \oplus y_\delta) = \phi_\gamma^\alpha x \oplus \phi_\gamma^\beta y$ is defined in L_γ, as desired.

($\Leftarrow$): Assume that $\exists \gamma \geq \alpha, \beta$ such that $\phi_\gamma^\alpha x \oplus \phi_\gamma^\beta y$ is defined in L_γ. Choose $\delta \geq \alpha, \beta, \gamma$, and set $x_\gamma := \phi_\gamma^\alpha x$ and $y_\gamma := \phi_\gamma^\beta y$. Then $x_\gamma \in \bar{x} \cap L_\gamma$ since $\phi_\delta^\gamma x_\gamma = \phi_\delta^\gamma \phi_\gamma^\alpha x = \phi_\delta^\alpha x$, and $y_\gamma \in \bar{y} \cap L_\gamma$ since $\phi_\delta^\gamma y_\gamma = \phi_\delta^\gamma \delta_\gamma^\beta y = \phi_\gamma^\beta y$. Moreover, $x_\gamma \oplus y_\gamma = \phi_\gamma^\alpha x \oplus \phi_\gamma^\beta y$ is defined in L_γ. Therefore, by Definition 2.6, $\bar{x} \oplus \bar{y}$ is defined in L. ∎

Note that Lemma 3.13 yields a $\oplus$-operation on $L = \{\bar{x} : x \in X = \cup_{\alpha \in D} L_\alpha\}$ by setting $\bar{x} \oplus \bar{y} := \overline{(\phi_\gamma^\alpha x \oplus \phi_\gamma^\beta y)}$ which is equivalent to the $\oplus$-operation given in Definition 2.6. It is easy to check that this $\oplus$-operation is well-defined.

Now using the above definition of $\oplus$ on L and using Lemma 3.13, it can be shown that the proof of Theorem 3.1 and the results leading to it can be adjusted to yield the following result, which is an improvement of Theorem 3.1. This improved result is equivalent to Theorem 2.7 of [18] for difference posets, since an effect algebra is equivalent to a difference poset [6].

3.14 Theorem. Direct limits exist in the category of effect algebras and effect algebra-morphisms. That is, if $(\{L_\alpha\}_{\alpha\in D}; \{\phi_\beta^\alpha\}_{\alpha\le\beta})$ is a directed system of effect algebras, where ϕ_β^α is a morphism for all $\alpha, \beta \in D$, $\alpha \le \beta$, then its direct limit exists (in the same category) and equals $(L, \{\phi^\alpha\}_{\alpha\in D})$, where L is as defined in Definition 2.6 and $\phi^\alpha : L_\alpha \to L$ is defined by $\phi^\alpha x_\alpha = \overline{x_\alpha}$.

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